Class TEST 1

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1 UGA

1. Number of ways in which 12 different things can be divided among 5 persons such that they can get $2, 2, 2, 3, 3$ things respectively, is
(a)$\frac{12!}{(3!)^2(2!)^3}$ (b)$\frac{12!5!}{(3!)^2(2!)^3}$ (c)$\frac{12!}{(3!)^2(2!)^4}$ (d)$\frac{12!5!}{(3!)^3(2!)^4}$

2. You are new to ISI, you are trying to find the Dean's office, but unfortunately you can't find any person to guide you, so you are randomly moving in the following manner: You start at (6,0) and in each step, you rotate by 60° anticlockwise in the road about the origin and move 7 units towards the positive X direction. If (p, q) represents the position of you after 2020 steps, then find $p^2 + q^2$.
(a)2023 (b)57 (c)59 (d)96

3. Indian Govt has decided to open two new institutes, one by merging ISI Kolkata, IIT Bombay in a complex plane and the another by merging ISI Kolkata and IISC. The real part is ISI Kolkata and the imaginary part is the other. So, the new part is like two complex numbers $z = x + iy$ and $w = u + iv$ be complex numbers on the unit circle $z^2 + w^2 = 1$. Then the number of ordered pairs $(z, w))$ is
(a)0 (b)4 (c)8 (d)∞

4. Let $A_1, A_2, A_3, \ldots\ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and Ai for $i = 1, 2, \ldots\ldots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 PA_2 \ldots\ldots PA_8$, is
(a)2^8 (b)2^9 (c)2^7 (d)N.O.T.

5. Let z be a complex number with nonzero imaginary part. If $\frac{2+3z+4z^2}{2-3z+4z^2}$ is a real number, then the value of $|z|^2$ is
(a)$\frac{1}{2}$ (b)$\frac{1}{3}$ (c)$\frac{1}{8}$ (d)N.O.T

6. The number of 4 digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is
(a)560 (b)569 (c)500 (d)280

7. Consider the following two subsets of $C : A = \{\frac{1}{z} : |z| = 2\}$ and $B = \{\frac{1}{z} : |z - 1| = 2\}$, then which is a circle?
(a)only A (b)only B (c)A and B (d)None

8. The system of inequalities $a_1 - a_2^2 \geq \frac{1}{4}, a_2 - a_3^2 \geq \frac{1}{4}, a_3 - a_4^2 \geq \frac{1}{4}, \cdots, a_n - a_1^2 \geq \frac{1}{4}$ has (Solutions in Reals)
(a)No soln (b) 1 soln (c) 2 soln (d) Infinite soln

9. How many ways are there to put one white and one black king on a chessboard so that they do not attack each other?
(a) 3610 (b)3611 (c)3612 (d)N.O.T

10. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}} + \frac{b^2}{\frac{c}{29}+\frac{a}{31}} + \frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a + b + c)$
(a) 37 (b)24 (c)14 (d)27

11. The number of triples (a, b, c) of positive integers satisfying the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 + \frac{2}{abc}$ is
(a)1 (b)3 (c)6 (d)0

12. Area enclosed by the curves $[|x|] + [|y|] = 3$ ([.] is GIF)is
(a)40 (b)10 (c)20 (d)30

13. If 3 numbers are chosen randomly from the set $\{1, 3, 3^2, \cdots, 3^n\}$ without replacement, then the probability that form an increasing geometric progression is
(a)$\frac{3}{2n}$ if n is odd (b)$\frac{3(n-1)}{2n(n+1)}$ if n is even (c)$\frac{3(n-1)}{2n(n+1)}$ if n is odd (d)$\frac{3n}{2(n^2-1)}$ if n is odd

14. If ω is complex cube root of unity and a, b, c are such that $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$ and $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega$, then the value of $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$ is
(a) -2 (b) -1 (c) 1 (d) 2

15. If $z + \frac{1}{z} = 1$ and $a = z^{2017} + \frac{1}{z^{2017}}$ and b is the last digit of the number $2^{2^n} - 1$, when the integer $n > 1$, then the value of $a^2 + b^2$ is
(a)23 (b)24 (c)26 (d)27

2 UGB

1. Given a polynomial $f(x)$ with integer coefficients whose value is divisible by 3 for three integers $k, k+1, k+2$, prove that $f(m)$ is divisible by 3 for all integers m.

Class TEST 2

Srijan Chattopadhyay

May 7, 2024

UGB

1. Let $a, b, c \geq 0$ be reals such that $abc = 1$. Find the maximum possible value of $\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca}$.

2. Let m; n and a be natural numbers and $p < a-1$ be a prime number. Prove that the polynomial $f(x) = x^m(x-a)^n + p$ cannot be factorised as $g(x)h(x)$ where g, h are non-constant polynomials with integer coefficients.

3. Consider the set $N_n = \{1, 2, \cdots, n\}$. Let S be a subset of N_n such that $\forall A, B \in S$, we have $A \nsubseteq B$. Now prove the following steps.

   (a) Let Π be a random permutation of N_n. Say, $S = \{A_1, \cdots, A_m\}$. Let E_i be the event that $\{\Pi(1), \Pi(2), \cdots, \Pi(|A_i|)\}$ is the **same set** as A_i. Prove that atmost one of E_i can happen out of $E_1, \cdots, E_m$.

   (b) Prove that, $P(E_i) = \frac{1}{\binom{n}{|A_i|}}$

   (c) Combining (a) and (b) prove that

   $$\sum_{A \in S} \frac{1}{\binom{n}{|A|}} \leq 1$$

4. Suppose that f(x) is a function satisfying $|f(m+n) - f(m)| \leq \frac{n}{m}$ for all rational numbers n and m. Show that, for all natural numbers k, $\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2}$.

5. If $S(n) = lim_{x \to 0} \sum_{r=1}^{n} \frac{\binom{n}{r} \sin rx \cos((n-r)x)}{x 2^n}$, find $S(2024)$.

6. Solve the question.

   Consider an $n \times n$ array of numbers:

   $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

   Suppose each row consists of the n numbers $1, 2, 3, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are $1, 2, 3, \ldots, n$ in some order.

7. Let T be the set of all tuples (a, b, c) such that they form a triangle. Find

   $$\sum_{(a,b,c) \in T} \frac{2^a}{3^b 5^c}$$

Class TEST 3

Srijan Chattopadhyay

9.17.2024

1 UGA

1. $f(x) = 3(3x^2 - 2x + 1)^2 - 2(3x^2 - 2x + 1) + 1 = x$ has how many solutions?
   (a)0 (b)1 (c)infinitely many (d)N.O.T.

2. Let $f : \mathbb{N} \to \mathbb{R}$ be such that $f(1) = 1$ and $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n)$, for $n \geq 2$, then $\frac{1}{2024f(2024)} =$
   (a)1 (b)2 (c)3 (d)Depends on n

3. $\lim_{n\to\infty} \cos(\frac{x}{2})\cos(\frac{x}{4})\cos(\frac{x}{8})\cdots\cos(\frac{x}{2^n}) =$
   (a) doesn't exist (b) 0 (c) 1 (d) depends on x

4. The value of $\sum_{r=1}^{\infty} \frac{(4r+5)5^{-r}}{r(5r+5)}$ =?
   (a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{25}$ (d) $\frac{2}{125}$

5. The total number of functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$, $\forall i < j$ is
   (a)35 (b)30 (c)50 (d)60

6. Find number of non negative integral solutions of $3x + y + z = 24$
   (a)110 (b)117 (c)114 (d)N.O.T

2 UGB

1. If $f : [0, \infty] \to \mathbb{N}$ is a right continuous function having left limits. Show that $\forall t \in [0, \infty)$, the set $\{y : f(x) = y, x \in [0, t]\}$ has a finite number of points, i.e., the range for each closed and bounded interval starting from 0 has a finite number of points. [Hint: You can use the fact that each bounded sequence has a convergent subsequence.]

2. Let $x_1, \cdots, x_{2024}$ be non negative real numbers with $\sum_{i=1}^{2024} x_i = 1$. Find, with proof, the minimum and maximum possible values of the following expression
$$\sum_{i=1}^{1012} x_i + \sum_{i=1013}^{2024} x_i^2$$

3. Let p be a prime and m be a positive integer for which $m < p$ and suppose decimal expansion of $\frac{m}{p}$ has period $2k$ for some positive integer k.
$$\frac{m}{p} = 0.ABABABABABAB\cdots$$
   where A, B are 2 distinct blocks of k digits, prove that $A + B = 10^k - 1$.

4. In a square $ABCD, E$ is on AB and F is on BC such that $\angle EDF = \angle CDF$. DE = 20 cm. Calculate $AE + CF$

Class TEST 4

Srijan Chattopadhyay

Nov 19, 2024, Time : 2 Hrs

UGB

1. Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

2. Let $f : (a, b) \to \mathbb{R}$ is continuously differentiable, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

4. Consider the sequence $(a_n)_{n \geqslant 1}$ defined by $a_1 = 1/2$ and $2n \cdot a_{n+1} = (n + 1)a_n$. Determine the general formula for a_n. Let $b_n = a_1 + a_2 + \cdots + a_n$. Prove that $\{b_n\} - \{b_{n+1}\} \neq \{b_{n+1}\} - \{b_{n+2}\}$.

5. Prove that for any prime $p \geq 3$, the number $\binom{2p-1}{p-1} - 1$ is divisible by p^2.

6. Let $0 < a < b$. Show that amongst the triangles with base a and perimeter $a + b$, the maximum area is obtained when the other two sides have length equal to $\frac{b}{2}$.

7. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be 2 permutations of the numbers $\{1, 2, \cdots, n\}$. Show that $\sum_{i=1}^{n} i(n + 1 - i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$.

8. 10 candidates participate for olympiad, which is organized around a table. There are 5 versions of the test and each candidate will receive exactly 1 version. No two consecutive candidates will get same version. How many ways there are to give the questions? [Hard Quesntion, try at the end]

Class TEST 5

Srijan Chattopadhyay

Dec 31, 2024, Time : 2 Hrs

UGB

1.
   - Show that $1^{2015} + 2^{2015} + 3^{2015} + 4^{2015} + 5^{2015} + 6^{2015}$ is divisible by 7.
   - Determine all pairs $(a, b) \in \mathbb{C} \times \mathbb{C}$ of complex numbers satisfying $|a| = |b| = 1$ and $a + b + a\bar{b} \in \mathbb{R}$.

2. Determine all the pairs of positive real numbers (a, b) with $a < b$ such that the following series

$$\sum_{k=1}^{\infty} \int_a^b \{x\}^k dx = \int_a^b \{x\} dx + \int_a^b \{x\}^2 dx + \int_a^b \{x\}^3 dx + \cdots$$

   is convergent and determine its value in function of a and b. $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. Assume you can swap the integral and summation.

3. Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy

$$f(x) = (x + 1)f(x^2),$$

   for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

4. Let $a_1, a_2, \ldots, a_{2024}$ be a permutation of $1, 2, \ldots, 2024$. Find the minimum possible value of

$$\sum_{i=1}^{2023} \Big((a_i + a_{i+1}) \Big(\frac{1}{a_i} + \frac{1}{a_{i+1}} \Big) + \frac{1}{a_i a_{i+1}} \Big)$$

5. In a sports tournament involving N teams, each team plays every other team exactly one. At the end of every match, the winning team gets 1 point and losing team gets 0 points. At the end of the tournament, the total points received by the individual teams are arranged in decreasing order as follows:

$$x_1 \geq x_2 \geq \cdots \geq x_N.$$

   Prove that for any $1 \leq k \leq N$,

$$\frac{N - k}{2} \leq x_k \leq N - \frac{k + 1}{2}$$

6. Let $f : [0, 1] \to (0, \infty)$ be a continuous function satisfying $\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0, 1)$ such that $\int_0^c f(t)dt = c - \frac{1}{2}$.

7. Consider the set of integers $\{1, 2, ..., 100\}$. Let $\{x_1, x_2, ..., x_{100}\}$ be some arbitrary arrangement of the integers $\{1, 2, ..., 100\}$, where all of the x_i are different. Find the smallest possible value of the sum $S = |x_2 - x_1| + |x_3 - x_2| + ... + |x_{100} - x_{99}| + |x_1 - x_{100}|$.

8. In a cyclic quadrilateral $ABCD$, the diagonals intersect at E. F and G are on chord AC and chord BD respectively such that $AF = BE$ and $DG = CE$. Prove that, A, G, F, D lie on the same circle.

Class TEST 6

Srijan Chattopadhyay

Feb 12, 2024, Time : 2 Hrs

UGB

1. Recall that the decimal expansion of a real number x is $I.d_1d_2\cdots$ if $x = I + \sum_{n=1}^{\infty} 10^{-n}d_n$ for some $I \in \mathbb{Z}$ and $d_1, d_2 \cdots \in \{0, 1, \cdots, 9\}$. Prove that for x, the decimal expansion $I.d_1d_2\cdots$ is not unique if and only if either $d_n = 0$ for all but finitely many n's or $d_n = 9$ for all but finitely many n's.

2. (a) Find all possible non-negative integer solution (x, y) of the following equation-

$$x! + 2^y = z!$$

   Note: $x! = x \cdot (x-1)!$ and $0! = 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

   (b) Consider an integrable function $f : \mathbb{R} \to \mathbb{R}$ such that $af(a) + bf(b) = 0$ when $ab = 1$. Find the value of the following integration: $\int_0^{\infty} f(x)\,dx$

3. For any positive integer n, define $f(n)$ to be the smallest positive integer that does not divide n. For example, $f(1) = 2$, $f(6) = 4$. Prove that for any positive integer n, either $f(f(n))$ or $f(f(f(n)))$ must be equal to 2.

4. For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

   where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

   giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

5. For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i.$$

   Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

6. Suppose $x_1, x_2, \cdots, x_{2n}$ are distinct points in the plane. Prove that you can join the points in such a way such that there will be exactly n^2 many edges (a direct line between two points is denoted as an edge) with no triangle in the whole picture.

7. Let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined as $x_n = \sin(2\pi n! e)$ for all $n \in \mathbb{N}$. Compute $\lim_{n \to \infty} x_n$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at 0. Define another function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Suppose that g is also differentiable at 0. Prove that

$$g'(0) = f'(0) = f(0) = g(0) = 0.$$

Class TEST 7

Srijan Chattopadhyay

Mar 17, 2024, Time : 2 Hrs

UGB

1.
   - If N is any odd positive integer and
   $$\delta_n = \int_0^N \binom{u}{n}\binom{N-u}{N-n} du,$$
   then the value of
   $$\sum_{n=0}^{(N-1)/2} N^{-1}(\delta_n + \delta_{N-n}) \text{ is}$$
   - Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let
$$J = \left\{ \frac{f(b) - f(a)}{b - a} : a, b \in I, a < b \right\}$$
Show that
   a) J is an interval.
   b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements.

3. If α is a root of the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients, $a_n \neq 0$, then prove that
$$|\alpha| \leq 1 + \max_{0 \leq k \leq n-1} \left| \frac{a_k}{a_n} \right|.$$

4. Prove that for any function $f : \mathbb{Q} \to \mathbb{Z}$, there exist $a, b, c \in \mathbb{Q}$ such that $a < b < c$, $f(b) \geq f(a)$ and $f(b) \geq f(c)$.

5. Let f be a continuous function on $[0, 1]$ such that for every $x \in [0, 1]$, we have $\int\limits_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\int\limits_0^1 f(t)^2 dt \geq \frac{1}{3}$.

6. In a sports tournament involving N teams, each team plays every other team exactly one. At the end of every match, the winning team gets 1 point and losing team gets 0 points. At the end of the tournament, the total points received by the individual teams are arranged in decreasing order as follows:
$$x_1 \geq x_2 \geq \cdots \geq x_N.$$
Prove that for any $1 \leq k \leq N$,
$$\frac{N-k}{2} \leq x_k \leq N - \frac{k+1}{2}$$

7. Find all natural numbers N for which $N(N-101)$ is a perfect square.

8. Let $m_1 < m_2 < \ldots m_{k-1} < m_k$ be k distinct positive integers such that their reciprocals are in arithmetic progression.

   1.Show that $k < m_1 + 2$. 2. Give an example of such a sequence of length k for any positive integer k.

Class TEST 8

Srijan Chattopadhyay

Mar 17, 2024, Time : 2 Hrs

UGA

1. The number of positive divisors of $2^{24} - 1$ is **(A)** 192 **(B)** 48 **(C)** 96 **(D)** 24.

2. The equation Re $z^2 = 0$ represents **(A)** a circle **(B)** a pair of straight lines **(C)** an ellipse **(D)** a parabola.

3. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and det $A^3 = 125$, then the values of α are **(A)** ± 1 **(B)** ± 2 **(C)** ± 3 **(D)** ± 5.

4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B and 3 from C is **(A)** exactly 1 **(B)** at most 1 **(C)** at most 2 **(D)** always 0.

5. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \frac{2023}{n}|a_n - a_{n-1}|$, $\forall n$. Then the sequence $\{a_n\}$ is **(A)** not Cauchy **(B)** Cauchy but not convergent **(C)** convergent **(D)** not bounded.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F be a primitive of f (i.e. $F' = f$). If

$$3x^2 F(x) = f(x) \text{ for all } x \in \mathbb{R} \text{ then } f(x) =$$

   (A) e^{x^3} **(B)** $3x^2 e^{x^3}$ **(C)** $x^2 e^{x^2}$ **(D)** $3xe^{x^3}$.

7. $1 \times 2 - 2 \times 3 + 3 \times 4 - 4 \times 5 + \cdots - (2022) \times (2023) =$ **(A)** $(-2)(1011)(1012)$ **(B)** $-(1011)(1012)$ **(C)** $(-4)(1011)(1012)$ **(D)** $2(1011)(1012)$.

8. The number of times the digit 7 is written while listing all integers from 1 to 1, 00, 000 is **(A)** 10^4 **(B)** $5(10)^4 - 1$ **(C)** 10^5 **(D)** $5(10)^4$.

9. The differential equation $y'^2 - (x + \sin x)y' + x \sin x = 0$, with $y(0) = 0$ has **(A)** unique solution **(B)** two solutions **(C)** no solution **(D)** four solutions.

10. Let A be a non-empty subset of real numbers and $f : A \to A$ be a function such that

$$f(f(x)) = x \text{ for all } x \in A.$$

    Then $f(x)$ is **A)** a bijection **B)** one-one but not onto **C)** onto but not one-one **D)** neither one-one nor onto.

11. If $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(xy)$ for all $x, y \in \mathbb{R}$ and $f\left(\frac{3}{4}\right) = \frac{3}{4}$, then $f\left(\frac{9}{16}\right) =$ **A)** $\frac{9}{16}$ **B)** 0 **C)** $\frac{3}{2}$ **D)** $\frac{3}{4}$.

12. The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \leq x \leq \frac{\pi}{2}$ is **A)** 2 **B)** $\frac{1}{2}$ **C)** 1 **D)** $\frac{3}{4}$.

13. The number of ordered pairs (m, n) of all integers satisfying

$$\frac{m}{12} = \frac{12}{n} \text{ is}$$

    A) 15 **B)** 30 **C)** 12 **D)** 10.

14. Suppose $2\log x + \log y = x - y$. Then the equation of the tangent line to the graph of this equation at the point $(1,1)$ is **A)** $x + 2y = 3$ **B)** $x - 2y = 3$ **C)** $2x + y = 3$ **D)** $2x - y = 3$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Then **A)** f is a 2π-periodic function **B)** f is a π-periodic function **C)** f is a 1-periodic function **D)** f is not a periodic function.

16. For how many integers a with $1 \leq a \leq 100$, a^a is a square? **A)** 50 **B)** 51 **C)** 55 **D)** 56.

17. $\lim_{x\to 0} x\left[\frac{1}{x}\right]$ **A)** 0 **B)** 1 **C)** -1 **D)** does not exist.

18. If α and β are the roots of $x^2 + 3x + 1$ then

$$\left(\frac{\alpha}{\beta+1}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2 \text{ equals}$$

A) 19 **B)** 18 **C)** 20 **D)** 17.

19. The equation $z^3 + iz - 1 = 0$ has **A)** no real root **B)** exactly one real root **C)** three real roots **D)** exactly two real roots.

20. How many five-digit positive integers that are divisible by 3 can be formed using the digits 0, 1, 2, 3, 4, and 5, without any of the digits getting repeated? **A)** 216 **B)** 96 **C)** 120 **D)** 625

21. If $I = \int_0^1 \frac{1}{1+x^8}\,dx$, then **A)** $I < \frac{1}{2}$ **B)** $I < \frac{\pi}{4}$ **C)** $I > \frac{\pi}{4}$ **D)** $I = \frac{\pi}{4}$.

22. Find a and b so that $y = ax + b$ is a tangent line to the curve $y = x^2 + 3x + 2$ at $x = 3$. **A)** $a = 9, b = -7$ **B)** $a = 3, b = -2$ **C)** $a = -9, b = 7$ **D)** $a = -3, b = 2$.

23. Suppose p is a prime number. The possible values of gcd of $p^3 + p^2 + p + 11$ and $p^2 + 1$ are **A)** 1, 2, 5 **B)** 2, 5, 10 **C)** 1, 5, 10 **D)** 1, 2, 10.

24. Consider all 2×2 matrices whose entries are distinct and belong to $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is **A)** 4! **B)** 0 **C)** negative **D)** odd.

25. If $x^3 - x + 1 = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$, then (a_0, a_1, a_2, a_3) equals **A.** $(1, -1, 0, 1)$ **B.** $(7, 6, 10, 1)$ **C.** $(7, 11, 12, 6)$ **D.** $(7, 11, 6, 1)$

26. Suppose $f(x)$ and $g(x)$ are real-valued differentiable functions such that $f'(x) \geq g'(x)$ for all x in $[0,1]$. Which of the following is necessarily true? **A.** $f(1) \geq g(1)$ **B.** $f(-g)$ has no maximum on $[0,1]$ **C.** $f(1) - g(1) \geq f(0) - g(0)$ **D.** $f + g$ is a non-decreasing function on $[0,1]$

27. The equation $x^4 + x^2 - 1 = 0$ has **A.** two positive and two negative roots **B.** one positive, one negative and two non-real roots **C.** one positive, one negative and two non-real roots **D.** no real root

28. Let X be a set and A, B, C be its subsets. Which of the following is necessarily true? **A.** $A \setminus (A \setminus B) = B$ **B.** $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$ **C.** $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$ **D.** $B \setminus (A \setminus B) = B$

29. For a real number x we let $[x]$ denote the largest integer not exceeding x. For a natural number n, let $a_n = \frac{[n\sqrt{2}]}{n}$. The limit $\lim_{n\to\infty} a_n$ **A.** equals 0 **B.** equals $\sqrt{2}$ **C.** equals $\sqrt{2}$ **D.** does not exist

30. Let n be any positive integer and $1 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 2n$, where each x_i is an integer. Which of the following must be true?

    (I) There is an i such that x_i is a square of an integer.

    (II) There is an i such that $x_{i+1} = x_i + 1$.

    (III) There is an i such that x_i is prime.

    A. I only **B.** II only **C.** I and II only **D.** II and III only

Class TEST 9

Srijan Chattopadhyay

Mar 27, 2024, Time : 2 Hrs

UGA

1. Consider $f(x) = x[x^2]$, where $[x^2]$ is the greatest integer less than or equal to x^2. Find the area of the region above X-axis and below $f(x)$,$1 \leq x \leq 10$. **(A)**2400 **(B)**2475 **(C)**3000 **(D)**N.O.T

2. Let $a_1, a2, \cdots$ be a sequence of natural numbers. Let (a, b) denote the greatest common divisor (gcd) of a and b. If $(a_m, a_n) = (m, n)$ for all $m \neq n$, then $a_{2025} =$ **(A)**1 **(B)** 2025 **(C)**2025^2 **(D)**N.O.T

3. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are **(A)** $k \leq 0$ **(B)** $k \geq -4$ **(C)** $k \geq 0$ or $k \leq -4$ **(D)** $-4 \leq k \leq 0$.

4. Let $M_2(\mathbb{Z}_2)$ denote the set of all 2×2 matrices with entries from $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the set of integers modulo 2. The function $f : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ given by $f(x) = x^2$ is **(A)** injective but not surjective **(B)** bijective **(C)** surjective but not injective **(D)** neither injective nor surjective.

5. Consider the sequence $4, 0, 4.1, 0, 4.11, 0, 4.111, 0, \cdots$. This sequence **(A)** converges to $4\frac{1}{9}$ **(B)** has no convergent subsequence **(C)** is unbounded **(D)** is not convergent and has supremum $4\frac{1}{9}$.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-2)|(x-2)(x-3)|$. The function f is **(A)** differentiable at $x = 2$ but not at $x = 3$ **(B)** differentiable at $x = 3$ but not at $x = 2$ **(C)** differentiable at $x = 2$ and $x = 3$ **(D)** neither differentiable at $x = 2$ nor at $x = 3$.

7. The equation $z^2 + \bar{z}^2 = 2$ represents the **(A)** parabola **(B)** pair of lines **(C)** hyperbola **(D)** ellipse.

8. The differential equation of the family of parabolas having their vertices at the origin and their foci on the X-axis is **(A)** $2xdy - ydx = 0$ **(B)** $xdy + ydx = 0$ **(C)** $2ydx - xdy = 0$ **(D)** $dy - xdx = 0$.

9. The number of solutions of the equation $\sqrt{1 - \sin x} = \cos x$ in $[0, 5\pi]$ is equal to **(A)** 3 **(B)** 6 **(C)** 8 **(D)** 11.

10. Consider $\triangle ABC$. Take 3 points on AB, 4 on BC and 5 on CA such that none of the points are vertices of $\triangle ABC$. The number of triangles that can be constructed using these points is **(A)** 60 **(B)** 205 **(C)** 145 **(D)** 120.

11. The number of primes p such that $p, p+10, p+14$ are all prime numbers is **(A)** 0 **(B)** 1 **(C)** 3 **(D)** infinitely many.

12. A relation R is defined on the set of positive integers as xRy if $2x + y \leq 5$. The relation R is **(A)** reflexive **(B)** symmetric **(C)** transitive **(D)** None of these.

13. Suppose $f : [0, 1] \to [0, 1]$ be a function. Then number of fixed points of f is **(A)** ≥ 1 **(B)** ≤ 1 **(C)** $= 1$ **(D)** N.O.T

14. If $p(x)$ is a non-constant polynomial, then

$$\lim_{k \to \infty} \frac{p(k+1)}{p(k)}$$

is equal to **(a)** 1 **(b)** 0 **(c)** -1 **(d)** the leading coefficient of $p(x)$.

15. The number of continuous functions f from $[-1,1]$ to $\mathbb{R}$ satisfying $(f(x))^2 = x^2$ for all $x \in [-1,1]$ is **(a)** 2 **(b)** 3 **(c)** 4 **(d)** infinite.

16. Let $q \in \mathbb{N}$. The number of elements in set $\{(\cos\frac{\pi}{q} + i\sin\frac{\pi}{q})^n | n \in \mathbb{N}\}$ is **(a)** 1 **(b)** q **(c)** infinite **(d)** $2q$.

17. If $f(x) = |x|^{\frac{3}{2}}$, $\forall x \in \mathbb{R}$, then at $x = 0$, **(a)** f is not continuous **(b)** f is continuous but not differentiable **(c)** f is differentiable but f' is not continuous **(d)** f is differentiable and f' is continuous.

18. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation $x^4 + x^3 + 1 = 0$, then the value of

$$(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_4)$$

is equal to **(a)** 19 **(b)** 16 **(c)** 15 **(d)** 20.

19. If A and B are 3×3 real matrices with rank $(AB) = 1$, then rank (BA) cannot be **(a)** 0 **(b)** 1 **(c)** 2 **(d)** 3.

20. The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is **(a)** 1 **(b)** 2 **(c)** 6 **(d)** 12.

21. If f is a one to one function from $[0,1]$ to $[0,1]$, then **(a)** f must be onto **(b)** f cannot be onto **(c)** $f([0,1])$ must contain a rational number **(d)** $f([0,1])$ must contain an irrational number.

22. There are 18 ways in which n identical balls can be grouped such that each group contains equal number of balls. Then the minimum value of n is **(a)** 120 **(b)** 180 **(c)** 160 **(d)** 90.

23. Let $A = (a_{ij})$ be 2025×2025 matrix, where $a_{ij} = \max\{i,j\}$ Then, $det(A)$ is **(A)** 0 **(B)** -1 **(C)** 2025 **(D)** -2025

24. Let n be a fixed positive integer. The value of k for which $\int_1^k x^{n-1}dx = \frac{1}{n}$ is **(a)** 0 **(b)** 2^n **(c)** $\left(\frac{2}{n}\right)^{\frac{1}{n}}$ **(d)** $2^{\frac{1}{n}}$.

25. Let $S = \{a,b,c\}, T = \{1,2\}$. If m denotes the number of one-one functions and n denotes the number of onto functions from S to T, then the values of m and n respectively are **(a)** 6,0 **(b)** 0,6 **(c)** 5,6 **(d)** 0,8.

26. For the equation $|x|^2 + |x| - 6 = 0$ **(a)** there is only one root. **(b)** the sum of the roots is -1. **(c)** the sum of the roots is 0. **(d)** the product of the roots is -6.

27. The value of $\lim_{n\to\infty} \frac{1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n!}{(n+1)!}$ **(a)** 1 **(b)** 2 **(c)** $\frac{1}{2}$ **(d)** does not exist.

28. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x)$ for every x. Then the graphs $y = f(x)$ and $y = g(x)$ **(a)** intersect exactly once. **(b)** intersect at most once. **(c)** do not intersect. **(d)** could intersect more than once.

29. The function $|x|^3$ is **(a)** differentiable twice but not thrice at 0. **(b)** not differentiable at 0. **(c)** three times differentiable at 0. **(d)** differentiable only once at 0.

30. The solution of $\frac{dy}{dx} = a^{x+y}$ is **(a)** $a^x - a^{-y} = c$ **(b)** $a^{-x} + a^{-y} = c$ **(c)** $a^{-x} - a^y = c$ **(d)** $a^x + a^{-y} = c$.

Class TEST 10

Srijan Chattopadhyay

Mar 27, 2024, Time : 2 Hrs

UGB

1. Suppose $f : \mathbb{R} \to [0, \infty)$. Define the following functions for $n \in \mathbb{N}$

$$s_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ 2^{-n}i & \text{if } 2^{-n}i \leq f(x) \leq 2^{-n}(i+1), i = 0, 1, 2, \cdots, n.2^n - 1 \end{cases}$$

   Prove that $s_1 \leq s_2 \leq s_3 \leq \cdots$. Also, prove that $\forall x$, $s_n(x) \to f(x)$ as $n \to \infty$.

2. Suppose $A_1, A_2, \cdots, A_n$ are vertices of a regular n-gon inscribed in a unit circle and P is any point on the unit circle. Prove that $\sum_{i=1}^{n} l(PA_i)^2$ is constant, where $l(PA_i)$ denotes the distance between P and A_i.

3. Define $\{x_n\}_{n \in \mathbb{N}}$ as $x_0 = 1$, and, $x_{n+1} = \ln(e^{x_n} - x_n)$. Prove that $\sum_{n=0}^{\infty} x_n$ converges, hence find the value.

4. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $0 \leq f'(x) \leq 1$ and $f(a) = 0$. Show that

$$3\left(\int_a^b f(x)^2 \, dx\right)^3 \geq \int_a^b f(x)^8 \, dx.$$

5. Suppose that the set $\{1, 2, \ldots, 1998\}$ is partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 999)$ in a manner that for each i, $|a_i - b_i|$ equals 1 or 6. Determine, with proof, the last digit of the sum

$$S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|.$$

6. Show that any positive integer n can be written as $n = a - b$ where a and b are positive integers having the same number of distinct prime divisors.

7. $n(> 1)$ lotus leaves are arranged in a circle. A frog jumps from a particular leaf from another under the following rule: It always moves clockwise. From starting it skips one leaf and then jumps to the next. After that it skips two leaves and jumps to the following. And the process continues. (Remember the frog might come back on a leaf twice or more.) Given that it reaches all leaves at least once. Show n cannot be odd.

8. Let u, v be real numbers. The minimum value of $\sqrt{u^2 + v^2} + \sqrt{(u-1)^2 + v^2} + \sqrt{u^2 + (v-1)^2} + \sqrt{(u-1)^2 + (v-1)^2}$ can be written as $\sqrt{n}$. Find the value of $10n$

Class TEST 11

Srijan Chattopadhyay

Apr 7, 2024, Time : 2 Hrs

UGB

1. Suppose $f : \mathbb{R} \to [0, \infty)$. For $\epsilon > 0$, define$f_\epsilon : \mathbb{R} \to [0, \infty)$ by

$$f_\epsilon(x) = n\epsilon \text{ when } n\epsilon \leq f(x) < (n+1)\epsilon$$

   Prove that $f_\epsilon(x) \to f(x)$ for all $x \in \mathbb{R}$ as $\epsilon \to 0$.

2. Suppose $f_1, f_2, \cdots, f_n$ are convex function from $\mathbb{R}$ to $\mathbb{R}$. Prove that $M(x) = \max\{f_1(x), \cdots, f_n(x)\}$ is also convex. [A function is called convex if for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$]. Can you say anything about the minimum function? 9 + 1

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

4. There are 130 students in a class. A test is conducted which consists of 5 questions. Suppose the teacher first checks a copy of one student, and the student gets distinct marks on all of the questions (suppose i on the ith question, $i = 1, \cdots, 5$). Suppose he gets $\{a, b, c, d, e\}$. Then, the teacher, being too lazy, takes a copy of a new student and randomly puts marks following the marks of the 1st student, also allowing repetition, i.e., for each question he/she can get anything within $\{a, b, c, d, e\}$ (i.e., (1,1,1,1,1), (1,5,5,5,5), etc are allowed.) Prove that there must exist two students who will get exact same set of marks. (i.e., (1,2,1,4,5) and (2,1,1,4,5) are considered asthe same set.)

5. $P(x)$ is a polynomial in x with non-negative integer coefficients. If $P(1) = 5$ and $P(P(1)) = 177$, what is the sum of all possible values of $P(10)$?

6. Given positive real numbers $a_1, a_2, ..., a_n$ with $a_1 + a_2 + ... + a_n = 1$. Prove that

$$\left(\frac{1}{a_1^2} - 1\right)\left(\frac{1}{a_2^2} - 1\right) ... \left(\frac{1}{a_n^2} - 1\right) \geqslant (n^2 - 1)^n$$

7. Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$

8. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.

Class TEST 12

Srijan Chattopadhyay

Apr 13, 2024, Time: 2 Hrs

UGA

1. Let α be a root of the equation $1+x^2+x^4=0$. Then the value of $\alpha^{1011}+\alpha^{2022}-\alpha^{3033}$ is equal to: (A)1 (B) α (C) $1+\alpha$ (D) $1+2\alpha$

2. Let $\arg(z)$ represent the principal argument of the complex number z. Then, $|z|=3$ and
$$\arg(z-1)-\arg(z+1)=\frac{\pi}{4}$$
intersect:

   (A) Exactly at one point (B) Exactly at two points (C) Nowhere (D) At infinitely many points.

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by
$$f(x)=(x-3)^{n_1}(x-5)^{n_2},\ n_1,\ n_2\in\mathbb{N}.$$
Then, which of the following is NOT true?

   (A) For $n_1=3$, $n_2=4$, there exists $\alpha\in(3,5)$ where f attains local maxima.

   (B) For $n_1=4$, $n_2=3$, there exists $\alpha\in(3,5)$ where f attains local minima.

   (C) For $n_1=3$, $n_2=5$, there exists $\alpha\in(3,5)$ where f attains local maxima.

   (D) For $n_1=4$, $n_2=6$, there exists $\alpha\in(3,5)$ where f attains local maxima.

4. The probability that a relation R from $\{x,y\}$ to $\{x,y\}$ is both symmetric and transitive, is equal to: (A) $\frac{5}{16}$ (B) $\frac{9}{16}$ (C) $\frac{11}{16}$ (D) $\frac{13}{16}$

5. Let $M=\begin{bmatrix}0 & -\alpha\\ \alpha & 0\end{bmatrix}$, where α is a non-zero real number and
$$N=\sum_{k=1}^{49}M^{2k}.$$
If $(I-M^2)N=-2I$, then the positive integral value of α is (A)1 (B)2 (C)3 (D)4

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by:
$$f(x)=\begin{cases}\max\limits_{t\le x}\{t^3-3t\}; & x\le 2\\ x^2+2x-6; & 2<x<3\\ [x-3]+9; & 3\le x\le 5\\ 2x+1; & x>5\end{cases}$$
where $[t]$ is the greatest integer less than or equal to t. Let m be the number of points where f is not differentiable and $I=\int_{-2}^{2}f(x)dx$. Then the ordered pair (m,I) is equal to:

   (A) $\left(3,\frac{27}{4}\right)$ (B) $\left(3,\frac{23}{4}\right)$ (C) $\left(4,\frac{27}{4}\right)$ (D) $\left(4,\frac{23}{4}\right)$

7. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and

$$a_{n+2} = 2a_{n+1} - a_n + 1 \quad \text{for all } n \geq 0.$$

Then,

$$\sum_{n=2}^{\infty} \frac{a_n}{7^n}$$

is equal to

(A) $\frac{6}{343}$ (B) $\frac{7}{216}$ (C) $\frac{8}{343}$ (D) $\frac{49}{216}$

8. Let $c,\, k \in \mathbb{R}$. If $f(x) = (c+1)x^2 + (1-c^2)x + 2k$ and

$$f(x+y) = f(x) + f(y) - xy, \quad \text{for all } x, y \in \mathbb{R},$$

then the value of $|2\,(f(1) + f(2) + f(3) + \cdots + f(20))|$ is equal to (A)3090 (B)3390 (C)3095 (D)3395

9. It is given that there are two sets of real numbers $A = \{a_1, a_2, \cdots, a_{100}\}$ and $B = \{b_1, b_2, \cdots, b_{50}\}$. If there is a mapping f from A to B such that every element in B has an inverse image and

$$f(a_1) \leq f(a_2) \leq \cdots \leq f(a_{100}),$$

then the number of such mappings is (A)$\binom{100}{50}$ (B)$\binom{99}{50}$ (C)$\binom{100}{49}$ (D)$\binom{99}{48}$

10. Let $M_n = \{0.a_1a_2\cdots a_n \mid a_i = 0 \text{ or } 1, 1 \leq i \leq n-1,\ a_n = 1\}$ be a set of binary fractions, T_n and S_n be the number and the sum of the elements in M_n respectively. Then

$$\lim_{n\to\infty} \frac{S_n}{T_n} =$$

(A)$\frac{1}{9}$ (B)$\frac{1}{10}$ (C)$\frac{1}{18}$ (D)$\frac{1}{20}$

11. Let $f(x)$ be a quadratic polynomial such that $f(-2) + f(3) = 0$. If one of the roots of $f(x) = 0$ is -1, then the sum of the roots of $f(x) = 0$ is equal to: (A)$\frac{11}{3}$ (B) $\frac{7}{3}$ (C) $\frac{13}{3}$ (D) $\frac{14}{3}$

12. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = \begin{cases} |x|^{1/8} |\sin\left(\frac{1}{x}\right)| \cos x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} e^x \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then, which one of the following is TRUE?

(A) f is continuous at $x = 0$, and g is NOT continuous at $x = 0$

(B) f is NOT continuous at $x = 0$, and g is continuous at $x = 0$

(C) f is continuous at $x = 0$, and g is continuous at $x = 0$

(D) f is NOT continuous at $x = 0$, and g is NOT continuous at $x = 0$

13. Let $x_1 = 2$ and $x_{n+1} = 2 + \dfrac{1}{2x_n}$ for all $n \in \mathbb{N}$. Then, which one of the following is TRUE?

(A) $x_{n+1} \geq \dfrac{4}{x_n}$ for all $n \in \mathbb{N}$, and (x_n) is a Cauchy sequence

(B) $x_{n+1} < \dfrac{4}{x_n}$ for some $n \in \mathbb{N}$, and (x_n) is a Cauchy sequence

(C) $x_{n+1} \geq \dfrac{4}{x_n}$ for all $n \in \mathbb{N}$, and (x_n) is NOT a Cauchy sequence

(D) $x_{n+1} < \dfrac{4}{x_n}$ for some $n \in \mathbb{N}$, and (x_n) is NOT a Cauchy sequence

14. Let $f(x) = x^4 + ax^3 + bx^2 + c$ be a polynomial with real coefficients such that $f(1) = -9$. Suppose that $i\sqrt{3}$ is a root of the equation

$$4x^3 + 3ax^2 + 2bx = 0,$$

where $i = \sqrt{-1}$. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are all the roots of the equation $f(x) = 0$, then

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2$$

is equal to (a)10 (b)20 (c)30 (d)40.

15. Let $k \in \mathbb{R}$. If

$$\lim_{x \to 0^+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6,$$

then the value of k is (A) 1 (B) 2 (C) 3 (D) 4

16. Let,

$$S = \left\{ \frac{a^2 + b^2 + c^2}{ab + bc + ca} : a, b, c \in \mathbb{R},\ ab + bc + ca \neq 0 \right\},$$

where $\mathbb{R}$ is the set of real numbers.
Then, S equals

a) $(-\infty, -1] \cup [1, \infty)$
b) $(-\infty, 0) \cup (0, \infty)$
c) $(-\infty, -1] \cup [2, \infty)$
d) $(-\infty, -2] \cup [1, \infty)$

17. The number of ordered pairs (x, y) of real numbers that satisfy the simultaneous equations $x + y^2 = x^2 + y = 12$ (A) 0 (B) 1 (C) 2 (D) 4

18. Let $p(x)$ be a polynomial such that $p(x) - p'(x) = x^n$, where n is a positive integer. Then p(0) equals (A) $n!$ (B) $(n-1)!$ (C) $\frac{1}{n!}$ (D) $\frac{1}{(n-1)!}$

19. There are 6 boxes labeled $B_1, B_2, \ldots, B_6$. In each trial, two fair dice D_1, D_2 are thrown. If D_1 shows j and D_2 shows k, then j balls are put into the box B_k. After n trials, what is the probability that B_1 contains at most one ball? (A) $\left(\frac{5^{n-1}}{6^{n-1}}\right) + \left(\frac{5^n}{6^n}\right)\left(\frac{1}{6}\right)$ (B) $\left(\frac{5^n}{6^n}\right) + \left(\frac{5^{n-1}}{6^{n-1}}\right)\left(\frac{1}{6}\right)$ (C) $\left(\frac{5^n}{6^n}\right) + n\left(\frac{5^{n-1}}{6^{n-1}}\right)\left(\frac{1}{6}\right)$ (D) $\left(\frac{5^n}{6^n}\right) + n\left(\frac{5^{n-1}}{6^{n-1}}\right)\left(\frac{1}{6^2}\right)$

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x) + \int_0^x t f(t)\, dt + x^2 = 0$$

for all $x \in \mathbb{R}$. Then (A) $\lim\limits_{x \to \infty} f(x) = 2$ (B) $\lim\limits_{x \to \infty} f(x) = -2$ (C) $f(x)$ has more than one point in common with the x-axis (D) $f(x)$ is an odd function

21. Let $C(\theta) = \sum_{n=0}^{\infty} \frac{\cos(n\theta)}{n!}$. Which of the following statements is FALSE? (A) $C(0) \cdot C(\pi) = 1$ (B) $C(0) + C(\pi) > 2$ (C) $C(\theta) > 0$ for all $\theta \in \mathbb{R}$ (D) $C'(\theta) \neq 0$ for all $\theta \in \mathbb{R}$

22. Let $a > 0$ be a real number. Then the limit

$$\lim_{x\to 2} \frac{a^x + a^{3-x} - (a^2+a)}{a^{3-x} - a^{x/2}} \text{ is}$$

(A) $2\log a$ (B) $-\frac{4}{3}a$ (C) $\frac{a^2+a}{2}$ (D) $\frac{2}{3}(1-a)$

23. Let $f(x) = \alpha x^2 - 2 + \frac{1}{x}$ where α is a real constant. The smallest α for which $f(x) \geq 0$ for all $x > 0$ is (A) $\frac{2^2}{3^3}$ (B) $\frac{2^3}{3^3}$ (C) $\frac{2^4}{3^3}$ (D) $\frac{2^5}{3^3}$

24. Suppose a, b, c are positive integers such that $2^a + 4^b + 8^c = 328$. Then

$$\frac{a+2b+3c}{abc}$$

is equal to (A) $\frac{1}{2}$ (B) $\frac{5}{8}$ (C) $\frac{17}{24}$ (D) $\frac{5}{6}$

25. The sides of a right-angled triangle are integers. The length of one of the sides is 12. The largest possible radius of the incircle of such a triangle is (A) 2 (B) 3 (C) 4 (D) 5

26. The value of the limit

$$\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{6/x^2} \text{ is}$$

(A) e (B) e^{-1} (C) $e^{-1/6}$ (D) e^6

27. Among all sectors of a fixed perimeter, choose the one with maximum area. Then the angle at the centre of this sector (i.e., the angle between the bounding radii) is (A) $\frac{\pi}{3}$ (B) $\frac{3}{2}$ (C) $\sqrt{3}$ (D) 2

28. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \max\{|x|, |x-1|, \ldots, |x-2n|\},$$

where n is a fixed natural number. Then

$$\int_0^{2n} f(x)\, dx \text{ is}$$

(A) n (B) n^2 (C) $3n$ (D) $3n^2$

29. If $p(x)$ is a cubic polynomial with $p(1) = 3$, $p(0) = 2$ and $p(-1) = 4$, then

$$\int_{-1}^{1} p(x)\, dx \text{ is}$$

(A) 2 (B) 3 (C) 4 (D) 5

30. Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Then a_{2023} can be (A) 1000 (B) 2000 (C) 3000 (D) 4000

Class TEST 13

Srijan Chattopadhyay

Apr 22, 2024, Time : 2 Hrs

UGB

1. Let m, n and p are real numbers such that $(m+n+p)\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{p}\right) = 1$. Find all possible values of
$$\frac{1}{(m+n+p)^{2023}} - \frac{1}{m^{2023}} - \frac{1}{n^{2023}} - \frac{1}{p^{2023}}.$$

2. Assume A and B are playing a game. They have n distinct numbers. They are asked to produce sets of maximum length n out of those n many numbers. A chooses all possible d sets out of n without repetition. Call the total number of distinct sets $A(n, d)$. B chooses n many numbers out of the given numbers with repetition. Call the total number of distinct sets to be $B(n)$. Whoever gets more number of sets, win. For what value of $d(n)$, $A(n, d)$ is maximized. Find
$$\lim_{n\to\infty} \frac{A(n, d(n))}{B(n)}$$
if it exists. So, for large enough n, who will win, given both play optimally within their strategy? Assume that
$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

3. Prove that all roots of $11z^{10} + 10iz^9 + 10iz - 11 = 0$ have unit modulus (or equivalent $|z| = 1$).

4. Suppose $F : \mathbb{R} \to \{0, 1\}$ is a function, i.e. F only takes two values 0 and 1. Suppose F is non decreasing, right continuous, and $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$. Prove that there exists some $x \in \mathbb{R}$ such that $F(y) = 1$ for all real $y \geq x$, and $F(y) = 0$ for all real $y < x$.

5. Consider a container of the shape obtained by revolving a segment (From $y = 0$ to $y = 5$) of the curve $x = e^{\frac{y}{5}}$ around the y-axis. The container is initially empty. Water is poured at a constant rate of $\pi\,\text{cm}^3$ into the container. Let $h(t)$ be the height of water inside the container at time t. Find $h(t)$ as a function of t.

6. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable and satisfy
$$f'(x) = -3f(x) + 6f(2x)$$
for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$. For $n \in \mathbb{N}$, define
$$\mu_n = \int_0^\infty x^n f(x)dx.$$
a. Express μ_n in terms of μ_0. b. Prove that the sequence $\frac{3^n \mu_n}{n!}$ always converges, and the the limit is 0 only if μ_0.

7. Let a,b,c be real numbers. Let z_1,z_2,z_3 be complex numbers such that $|z_k| = 1$ $(k = 1, 2, 3)$ and $\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1} = 1$ Find $|az_1 + bz_2 + cz_3|$.

8. If nonnegative reals $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \le k < j \le n} \sqrt{\frac{k}{j}} x_k x_j = 1$$

what are the minimum and maximum values of $\sum_{i=1}^{n} x_i$?

Class TEST 14

Srijan Chattopadhyay

Apr 26, 2024, Time: 2 Hrs

UGA

1. Consider all lines which meet the graph of $y = 2x^4 + 7x^3 + 3x - 5$ in four distinct points, say $(x_i, y_i), i = 1, 2, 3, 4$. Then Suppose
$$\frac{x_1 + x_2 + x_3 + x_4}{4} = am + b,$$
where m is the slope of the line. Then, (A) $a > 0$ (B) $a < 0$ (C) a = 0 (D) Can't be expressed as linear function of m.

2. Let a, b, c be real number greater than 1. Let
$$S = \log_a bc + \log_b ca + \log_c ab$$
Then the minimum possible value of S is (A) 3 (B) 6 (C) 9 (D) 12

3. Suppose $P(x)$ is an integer polynomial such that $P(1) = 7$ and $P(n)$ is prime for all $n \in \mathbb{N}$. Then, $P(2025) =$ (A) 7^{2025} (B) 2025×7 (C) 7 (D) N.O.T

4. Let $f(x)$ be a function such that $f(1) = 1$ and for $x \geq 1$
$$f'(x) = \frac{1}{x^2 + f(x)^2}$$
Then, $\lim_{x\to\infty} f(x)$ (A) exists and is less than $1 + \frac{\pi}{4}$ (B) exists and is greater than $1 + \frac{\pi}{4}$ (C) may not exist (D) exists but nothing more can be said.

5. Let $x_1 > 0$. For $n \in \mathbb{N}$, define $x_{n+1} = x_n + 4$. If
$$\lim_{n\to\infty} \left(\frac{1}{x_2 x_3} + \frac{1}{x_3 x_4} + \cdots + \frac{1}{x_{n+1} x_{n+2}} \right) = \frac{1}{24}$$
Then the value of x_1 is equal to (A) 1 (B) 2 (C) 3 (D) 8

6. For $n \in \mathbb{N}$, define x_n and y_n by $x_n = (-1)^n \cos \frac{1}{n}$ and $y_n = \sum_{k=1}^{m} \frac{1}{n+k}$.
Then, which one of the following is TRUE?
    - (A) $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ does NOT converge
    - (B) $\sum_{n=1}^{\infty} x_n$ does NOT converge, and $\sum_{n=1}^{\infty} y_n$ converges
    - (C) $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ converges
    - (D) $\sum_{n=1}^{\infty} x_n$ does NOT converge, and $\sum_{n=1}^{\infty} y_n$ does NOT converge

7. Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. $P(0) + Q(0) =$? (A) 116 (B) 120 (C) 124 (D) N.O.T

8. The roots of $x^3 + 2x^2 - x + 3$ are p, q, and r. What is the value of

$$(p^2 + 4)(q^2 + 4)(r^2 + 4)?$$

(A) 64 **(B)** 75 **(C)** 100 **(D)** 125

9. The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

(A) 318 **(B)** 319 **(C)** 320 **(D)** 321

10. A graph is *symmetric* about a line if the graph remains unchanged after reflection in that line. For how many quadruples of integers (a, b, c, d), where $|a|, |b|, |c|, |d| \leq 5$ and c and d are not both 0, is the graph of

$$y = \frac{ax + b}{cx + d}$$

symmetric about the line $y = x$?

(A) 1282 **(B)** 1292 **(C)** 1310 **(D)** 1320

11. There is a unique sequence of integers $a_1, a_2, \cdots a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

(A) -2023 **(B)** -2022 **(C)** -1 **(D)** 1

12. When the roots of the polynomial

$$P(x) = \prod_{i=1}^{10} (x - i)^i$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of those intervals is $P(x)$ positive?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6

13. Suppose a continuous function $f : [0, \infty) \to \mathbb{R}$ satisfies $f(x) = 2\int_0^x tf(t)\,dt + 1$ for all $x \geq 0$. Then $f(1)$ equals (A)e (B)e^2 (C)e^4 (D)e^6

14. Let $a > 0$, $a \neq 1$. Then the set S of all positive real numbers b satisfying

$$(1 + a^2)(1 + b^2) = 4ab$$

is

(A) an empty set

(B) a singleton set

(C) a finite set containing more than one element

(D) $(0, \infty)$

15. Let $p(x) = x^2 + ax + b$ have two distinct real roots, where a, b are real numbers. Define $g(x) = p(x^3)$ for all real numbers x. Then which of the following statements are true?

I. g has exactly two distinct real roots

II. g can have more than two distinct real roots

III. There exists a real number α such that $g(x) \geq \alpha$ for all real x

(A) Only I

(B) Only I and III
(C) Only II
(D) Only II and III

16. Let ABC be a triangle in which $AB = BC$. Let X be a point on AB such that $AX : XB = AB : AX$. If $AC = AX$, then the measure of $\angle ABC$ equals

(A) $18°$
(B) $36°$
(C) $54°$
(D) $72°$

17. What is the sum of all natural numbers n such that the product of the digits of n (in base 10) is equal to $n^2 - 10n - 36$?

(A) 12
(B) 13
(C) 124
(D) 2612

18. What is the area of the region in the coordinate plane defined by the inequality

$$||x| - 1| + ||y| - 1| \leq 1?$$

(A) 4 **(B)** 8 **(C)** 10 **(D)** 12

19. Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{127}{1024}$ **(D)** $\frac{511}{1024}$

20. A real-valued function f has the property that for all real numbers a and b,

$$f(a + b) + f(a - b) = 2f(a)f(b).$$

Which one of the following cannot be the value of $f(1)$? **(A)** 1 **(B)** -1 **(C)** 2 **(D)** -2

21. For complex numbers $u = a + bi$ and $v = c + di$, define the binary operation $\otimes$ by

$$u \otimes v = ac + bdi.$$

Suppose z is a complex number such that $z \otimes z = z^2 + 40$. What is $|z|$?

(A) $\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $2\sqrt{6}$ **(D)** 6

22. Let $f(x) = x|\sin x|$, $x \in \mathbb{R}$. Then

(A) f is differentiable for all x, except at $x = \eta\pi$, $\eta = 1, 2, 3, \ldots$
(B) f is differentiable for all x, except at $x = \eta\pi$, $\eta = \pm1, \pm2, \pm3, \ldots$
(C) f is differentiable for all x, except at $x = \eta\pi$, $\eta = 0, 1, 2, 3, \ldots$
(D) f is differentiable for all x, except at $x = \eta\pi$, $\eta = 0, \pm1, \pm2, \pm3, \ldots$

23. Let $\ln x$ denote the logarithm of x with respect to the base e. Let $S \subset \mathbb{R}$ be the set of all points where the function $\ln(x^2 - 1)$ is well-defined. Then the number of functions $f\colon S \to \mathbb{R}$ that are differentiable, satisfy

$$f'(x) = \ln(x^2 - 1) \quad \text{for all } x \in S$$

and $f(2) = 0$, is

(A) 0
(B) 1
(C) 2
(D) infinite

24. Let S be the set of real numbers p such that there is no nonzero continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_0^x f(t)\,dt = pf(x) \quad \text{for all } x \in \mathbb{R}.$$

Then S is

(A) the empty set
(B) the set of all rational numbers
(C) the set of all irrational numbers
(D) the whole set $\mathbb{R}$

25. Suppose A is a 3×3 matrix consisting of integer entries that are chosen at random from the set $\{-1000, -999, \ldots, 999, 1000\}$. Let P be the probability that either $A^2 = -I$ or A is diagonal, where I is the 3×3 identity matrix. Then

(A) $P < \frac{1}{10^{18}}$
(B) $P = \frac{1}{10^{18}}$
(C) $\frac{5^2}{10^{18}} \leq P \leq \frac{5^3}{10^{18}}$
(D) $P \geq \frac{5^4}{10^{18}}$

26. Suppose the limit $L = \lim\limits_{n\to\infty} \sqrt{n} \int\limits_0^1 \frac{1}{(1+x^2)^n}\,dx$ exists and is larger than $\frac{1}{2}$. Then

(A) $\frac{1}{2} < L < 2$
(B) $2 < L < 4$
(C) $3 < L < 4$
(D) $L \geq 4$

27. Suppose z is any root of $11z^8 + 20iz^7 + 10iz - 22 = 0$, where $i = \sqrt{-1}$. Then $S = |z|^2 + |z| + 1$ satisfies

(A) $S \leq 3$
(B) $3 < S < 7$
(C) $7 \leq S < 13$
(D) $S \geq 13$

28. Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

$$\lim_{n\to\infty} \frac{M(n)}{n} =$$

(A)$\frac{3-\sqrt{5}}{2}$ (B)$\frac{3+\sqrt{5}}{2}$ (C)1 (D)0

29. Let D_n be the determinant of order n of which the element in the i-th row and the j-th column is $|i-j|$. Then D_n is equal to
(A)$(-1)^{n-1}(n-1)2^{n-2}$ (B)$(-1)^n(n-1)2^{n-1}$ (C)$(-1)^{n-1}(n-1)2^{2n-2}$ (D)N.O.T

30. The inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds only for

(A)$\mathbb{N}$ (B)$\mathbb{Q}$ (C)$\mathbb{N} \cup \{0\}$ (D)$\mathbb{R}$